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CALCULATION OF FAMILIES OF STATIONARY FILTRATION CONVECTION REGIMES IN A NARROW CONTAINER

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A plane Darcy filtration convection problem for rectangular containers elongated in the vertical direction is considered. By the spectral-difference method, which preserved cosymmetry of the initial problem, evolution of families of stationary regimes from the onset of instability on the primary family till the collision of the families is studied.

Key words: *filtration convection, cosymmetry, families of stationary solutions, spectral-difference method.*

Introduction. Previous studies of incompressible-liquid convection in a porous medium (Darcy model) revealed simultaneous existence of an infinite number of stationary regimes [1]. This phenomenon was explained based on the cosymmetry theory [2, 3] and studied in natural [4, 5] and numerical experiments [6–11]. A continuous family of stationary convection regimes differing in their spectral characteristics arises after the mechanical-equilibrium state loses stability. With increasing heating intensity, some equilibria on the family become unstable. It follows from calculations based on the Galerkin method [6, 7] that, depending on geometric characteristics, such a transition can result both from monotonic and oscillatory instability. This implies appearance of either a positive real eigenvalue or a pair of complex numbers with positive real parts in the equilibrium spectrum. In addition, the calculations [7–9] revealed collisions of families of stationary regimes and appearance of periodic and random motions.

In the present work, using the spectral-difference method [11], we study the evolution of families of stationary convective regimes for rectangular containers whose height is larger than their width. The calculations were performed for disretizations yielding systems of ordinary differential equations with the number of variables ranging from several hundreds to a thousand. The emergence of instability on the primary family, the loss of stability of all equilibrium states of the family, and the collisions of families are analyzed.

Statement of the Problem. In the plane statement, we consider the problem about heating from below of a rectangular container filled with a porous medium saturated with a liquid. For a liquid obeying the Darcy law, the equations in dimensionless variables have the form

$$\theta_t = \Delta\theta + \lambda\psi_x + J(\theta, \psi); \qquad (1)$$

$$\Delta\psi - \theta_x = 0. \qquad (2)$$

Here θ is the deviation of temperature from the value corresponding to the equilibrium state, ψ is the stream function, $J(\theta, \psi) = \theta_x\psi_y - \theta_y\psi_x$ is the Jacobian, t is the time, and x and y are the plane Cartesian coordinates. The filtration Rayleigh number is defined by the formula $\lambda = \alpha gAkl/(\chi\nu)$, where α is the volume expansion, g is the free-fall acceleration, A is the typical temperature difference, k is the permeability, l is the characteristic length, χ is the thermal diffusivity, and ν is the kinematic viscosity. At the boundary of the region $\Omega = [0, a] \times [0, b]$, the following first-kind boundary conditions are posed:

$$\theta = 0, \qquad \psi = 0. \qquad (3)$$

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The initial condition for system (1), (2) has the form

$$\theta(x, y, 0) = \theta_0(x, y), \tag{4}$$

where $\theta_0(x, y)$ is a function defined in Ω. No initial condition for ψ is posed, since the stream function can be found from temperature by solving the boundary-value problem (2), (3). System (1)–(3) is globally solvable and dissipative, and the cosymmetry depends on the stream function ψ [2, 3]. Note that Eqs. (1) and (2) are invariant with respect to the transformations

$$R_x: \qquad \{x, y, \theta, \psi\} \mapsto \{a - x, y, \theta, -\psi\}; \tag{5}$$

$$R_y: \qquad \{x, y, \theta, \psi\} \mapsto \{x, b - y, -\theta, -\psi\}. \tag{6}$$

The zero equilibrium $\theta = \psi = 0$ for system (1)–(4) exists for all values of the parameter λ, and the eigenvalues of the corresponding spectral problem are given by the formula $\lambda_{nm} = 4\pi^2(n^2/a^2 + m^2/b^2)$, where m and n are integer numbers. Yudovich showed [3] the first critical value λ_{11} to be twofold for an arbitrary region and, as the parameter λ passes through λ_{11}, a family of stationary regimes with a variable spectrum (primary family) branches off from the state of rest. Further, each transition of λ through the next critical value λ_{nm} $(n + m > 2)$ leads to branching-off of a new family of unsteady stationary regimes from zero equilibrium.

Solution Method. To solve the problem, we use the spectral-difference method based on Galerkin expansions in the coordinate y and difference approximations along the coordinate x [11]. The solution is sought in the form

$$\{\theta, \psi\} = \sum_{j=1}^{m} \{\theta_j(x, t), \psi_j(x, t)\} \sin \frac{\pi j y}{b}. \tag{7}$$

After insertion of (7) into Eqs. (1) and (2) and projecting, we obtain the system

$$\begin{gathered} \dot{\theta}_j = \theta_j'' - c_j\theta_j + \lambda\psi_j' - J_j, \qquad J_j = \frac{2\pi}{b}\Big(\sum_{i=1}^{m-j} \chi_{j,i}^1 + \sum_{i=1}^{j-1} \chi_{j,i}^2\Big), \\ \psi_j'' - c_j\psi_j - \theta_j' = 0, \qquad j = 1, \ldots, m. \end{gathered} \tag{8}$$

Here and below, the prime denotes differentiation with respect to x and the dot denotes differentiation with respect to t; $c_j = j^2\pi^2/b^2$, and the quantities $\chi_{j,i}^1$ and $\chi_{j,i}^2$ can be represented as

$$\chi_{j,i}^1 = ((2i + j)/2)(D_s(\theta_{i+j}, \psi_i) - D_s(\theta_i, \psi_{i+j})) - (j/2)(D_a(\theta_{i+j}, \psi_i) + D_a(\theta_i, \psi_{i+j})),$$

$$\chi_{j,i}^2 = ((j - i)/2)(D_s(\theta_i, \psi_{j-i}) + D_a(\theta_{i+j}, \psi_i) - D_s(\theta_{j-i}, \psi_i) + D_a(\theta_{j-i}, \psi_i)),$$

where D_a and D_s are differential operators:

$$D_a(\theta, \psi) = \theta'\psi - \theta\psi', \qquad D_s(\theta, \psi) = \theta'\psi + \theta\psi'. \tag{9}$$

The boundary conditions for problem (8) are written as

$$\theta_j(t, 0) = \theta_j(t, a) = 0, \qquad \psi_j(t, 0) = \psi_j(t, a) = 0, \qquad j = 1, \ldots, m. \tag{10}$$

To approximate Eqs. (8) along the variable x, we use the finite-difference method of second-order accuracy. In the segment $[0, a]$, we introduce the grid $\omega = \{x_k\colon\ x_k = kh,\ k = 0, \ldots, n, \text{ and } h = a/(n + 1)\}$. Here and below, we use the designations $\theta_{j,k} = \theta_j(x_k, t)$, $\psi_{j,k} = \psi_j(x_k, t)$, and $J_{j,k} = J_j(x_k, t)$.

The first and second derivatives of the linear part of Eqs. (8) are approximated with central-difference relations. As a result, we obtain the following system of ordinary differential equations:

$$\begin{gathered} \dot{\theta}_{j,k} = \frac{\theta_{j,k+1} - 2\theta_{j,k} + \theta_{j,k-1}}{h^2} - c_j\theta_{j,k} + \lambda\,\frac{\psi_{j,k+1} - \psi_{j,k-1}}{2h} - J_{j,k}, \\ \frac{\psi_{j,k+1} - 2\psi_{j,k} + \psi_{j,k-1}}{h^2} - c_j\psi_{j,k} - \frac{\theta_{j,k+1} - \theta_{j,k-1}}{2h} = 0. \end{gathered} \tag{11}$$

From the boundary conditions (10), we find the grid functions on the boundaries:

$$\theta_{j,0} = \theta_{j,n} = 0, \qquad \psi_{j,0} = \psi_{j,n} = 0. \tag{12}$$

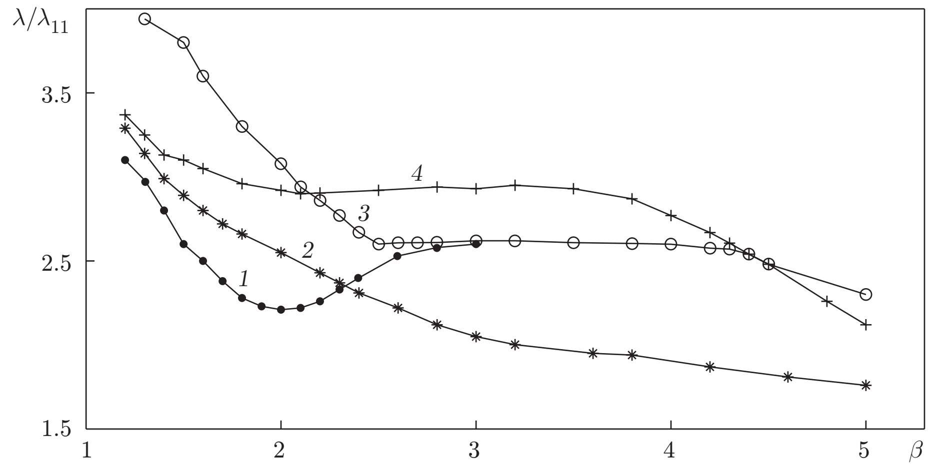

Fig. 1. Critical values versus on the container height: emergence of monotonic instability on the family (1), emergence of oscillatory instability (2), instability of the family as a whole (3), and collision of the primary family with the secondary-family branch (4).

Approximations of the operators D_a and D_s [see (9)] preserving the cosymmetry for problem (11), (12) were constructed in [11]:

$$d_{a,k}(\theta,\psi) = \frac{\theta_{k+1}-\theta_{k-1}}{2h}\,\psi_k - \theta_k\,\frac{\psi_{k+1}-\psi_{k-1}}{2h},$$

$$d_{s,k}(\theta,\psi) = \frac{2\theta_{k+1}\psi_{k+1} + \psi_k(\theta_{k+1}-\theta_{k-1}) + \theta_k(\psi_{k+1}-\psi_{k-1}) - 2\theta_{k-1}\psi_{k-1}}{6h}.$$

The initial condition (4) yields the initial temperatures

$$\theta_{j,k} = \int\limits_D \theta_0(x_k,y)\sin\frac{\pi j y}{b}, \qquad j=1,\ldots,m, \quad k=0,\ldots,n.$$

The resultant system of ordinary differential equations was integrated by the Runge–Kutta method. As was found in [3], for λ only insignificantly exceeding λ_{11}, all equilibria on the family are stable. Thus, starting from a vicinity of unstable zero equilibrium and integrating the corresponding system of ordinary differential equations until the solution attains its steady-state form, one can obtain some stable equilibrium state of the family. Afterwards, the family can be calculated using the algorithm [6, 8, 10]; in this algorithm, the linearization matrix is obtained numerically, and its kernel is found by the SVD-expansion method. To refine the solution for the equilibrium state in the vicinity of the family, the Newton method can be used, and the value at a next point on the family can be calculated by the Adams extrapolation method.

Calculation of Families of Stationary Regimes. Below, we present calculation results for families of stationary convection regimes in narrow containers of relative height $\beta = b/a > 1$. The width of the containers was fixed: $a = 1$, and the height b was a varied parameter. The study was performed for filtration Rayleigh numbers from the moment of emergence of the primary stable family to the moment of the collision of the families.

The state of rest $\theta = \psi = 0$ is globally stable if temperature gradients are small (the Rayleigh parameter is $\lambda < \lambda_{11}$); as we pass through a critical value, there arises a continuous family of stationary convection regimes that inherits the zero-equilibrium stability [2, 3]. The spectrum of each equilibrium contains a zero value corresponding to the neutral direction along the family; stability or instability of the equilibrium of the family can be established from a stability analysis performed on the manifold transverse to the family.

At small supercriticalities, i.e., at parameters λ slightly higher than the critical value λ_{11}, the primary family is perfectly stable. As the parameter increases, the family first grows, and at $\lambda = \lambda_u$ new equilibria (stationary convection regimes) with a neutral spectrum on the transverse manifold emerge on it. The total number of such regimes depends on the particular container geometry. Then, these equilibria lose stability (the second transition), and arcs of unstable regimes appear on the family.

If $\lambda_{12} < \lambda_u$, then a second family that consists of unstable regimes and belongs to the invariant subspace — consequence of the discrete symmetry of R_y [see (6)] — branches off. With increasing Rayleigh number, as a result

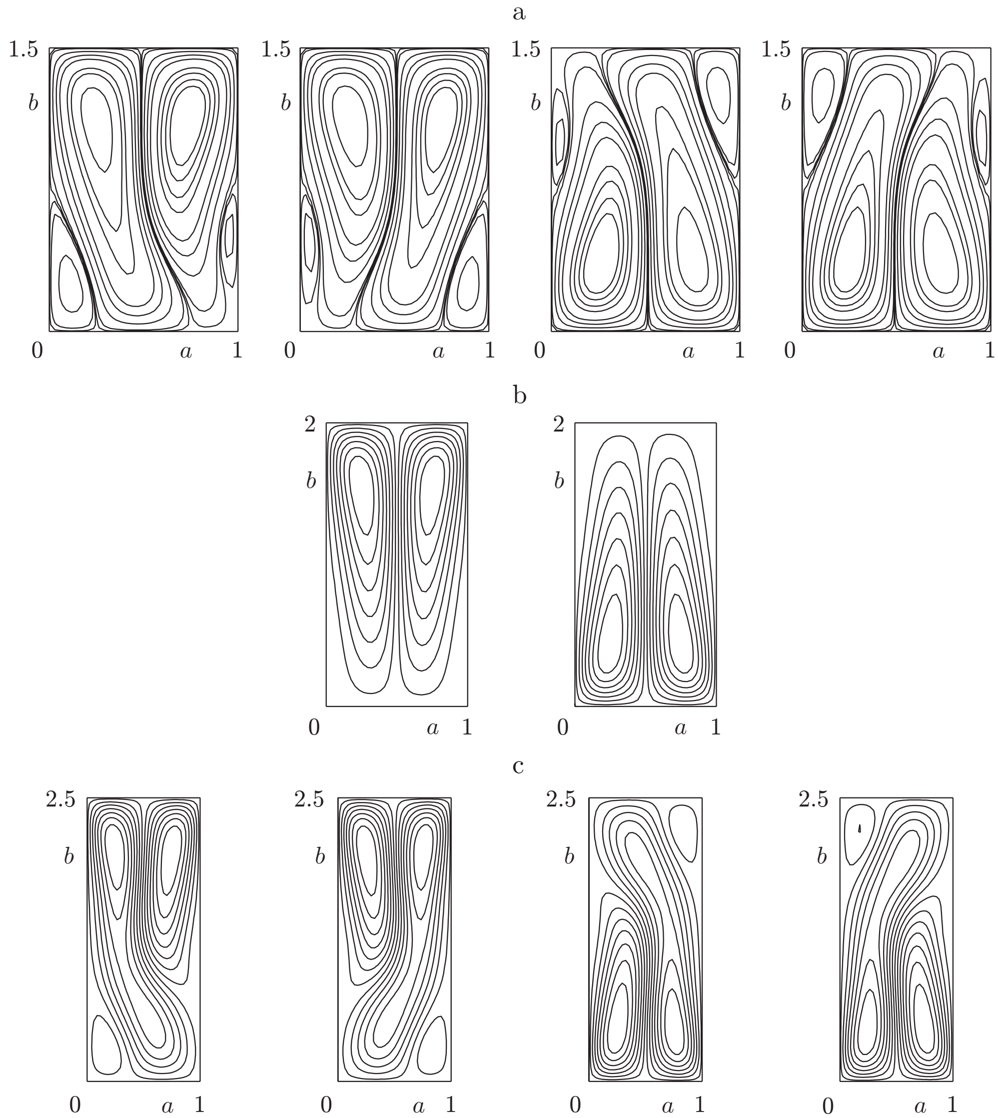

Fig. 2. Stream functions of stationary regimes that lose stability first for various relative container heights: $\beta = 1.5$ (a), 2 (b), and 2.5 (c).

of internal bifurcation, on this family there arises a branch that does not lie on the invariant subspace [8]; as λ further grows, this branch collides with the primary family. As a result, reconnection of the branches occurs, and new closed curves of families of stationary regimes emerge in the phase space.

Depending on the container height, two scenarios are possible. If $2.1 < \beta < 4.4$, then the primary family first becomes absolutely unstable ($\lambda = \lambda_o$), and then it collides with the second family ($\lambda = \lambda_c$). In containers with $1.0 < \beta \leqslant 2.1$ and $\beta \geqslant 4.4$, a collision of the primary family, which consists of stable and unstable stationary regimes, with the second, absolutely unstable family is observed. In all cases, the calculations yield three closed curves formed by equilibria: two additional (small) families and a family formed by merging of almost all parts of the primary families. For almost square containers, these additional families initially consist of stable and unstable regimes. It should be noted that, for $\beta < 1.34$, these families survive as the Rayleigh parameter increases, and the additional families for higher containers shrink into points and vanish.

The calculated critical values λ_u, λ_c, and λ_o for various β are shown in Fig. 1 (the values of λ are normalized by the critical value of the first transition λ_{11}). Curves 1 and 2 correspond to the emergence of monotonic and oscillatory instability on the family, curve 3 to the loss of stability of the primary family as a whole ($\lambda = \lambda_o$), and curve 4 to the critical values of the collision between the primary and secondary families ($\lambda = \lambda_c$). These data were

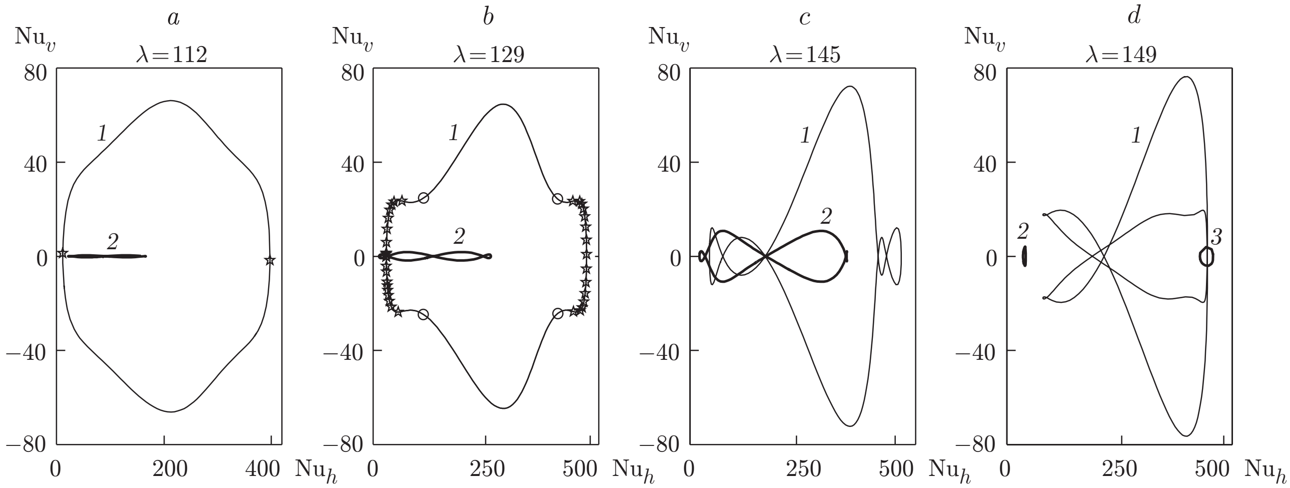

Fig. 3. Families of stationary regimes ($\beta = 2$) prior to the collision (a–c) (1 is a primary family and 2 is a secondary family) and after the collision (d) (1 is a merged family and 2 and 3 are small families).

obtained for the discretization with $n \times m = 16 \times 16$; to check the solutions, computations with $n \times m = 24 \times 16$, 24×24, and 30×30 were also performed.

Emergence of Instability on the Family. In [6, 7], for Galerkin models, an oscillatory instability for a narrow container with $b/a = 2.5$ was established. It follows from the results of the present study that, in the case of narrow containers with $\beta < 2.3$, the emergence of unstable arcs on the family of stationary regimes results from monotonic instability. With varying β, a continuous transition from monotonic instability to oscillatory instability is observed, and there is a value of β that corresponds to loss of stability at six points of the family: four points exhibit oscillatory instability and two points, monotonic instability. For the discretization with $n \times m = 16 \times 16$, the emergence of instability at six points was obtained for $\beta \approx 2.3$; for finer discretizations, the corresponding value of β insignificantly increases.

Instability on the family arises simultaneously at an even number of points since the stationary problem (1), (2) is invariant with respect to discrete symmetries (5) and (6). Figure 2 shows the stream functions of regimes that lose stability first for several values of β. In particular, monotonic instability is observed at four points for containers with $1.0 < \beta < 1.8$ and at two points if $1.8 \leqslant \beta < 2.3$. Note that monotonic instability at two points corresponds to regimes with the minimum or maximum heat flux through the vertical midsection. The second transition resulting from oscillatory instability can occur at two ($3.9 \leqslant \beta < 4.8$) or four points ($2.3 \leqslant \beta < 3.9$, $\beta > 4.8$). For $\beta < 3.3$, the equilibria on the family become unstable owing to monotonic or oscillatory instability, and for $\beta > 3.3$, only oscillatory instability causes them.

Evolution Scenarios of Families. In Fig. 3, the families of stationary regimes are shown in the coordinates Nu_h and Nu_v [6]:

$$\mathrm{Nu}_h = \int_0^b \frac{\partial\theta}{\partial x}\Big|_{x=a/2} dy, \qquad \mathrm{Nu}_v = \int_0^a \frac{\partial\theta}{\partial y}\Big|_{y=0} dx$$

[the asterisks (or circles) show the regimes that correspond to emergence of monotonic (or oscillatory) instability].

The primary family of equilibria for a container with $\beta = 2$ branches off at $\lambda = 50.6$, and the secondary family, at $\lambda = 81.6$. With increasing λ, the primary family undergoes deformation, and at $\lambda = \lambda_u = 112$ (see Fig. 3a) two points corresponding to monotonic instability appear on this family. Then, two arcs that correspond to unstable equilibria appear on the family, and four regimes that lost their stability in an oscillatory manner emerge at $\lambda = 129$ (Fig. 3b). The position of two families prior to the collision is shown in Fig. 3c, the major part of the primary family being already unstable (stable are only small arcs corresponding to the maximum and minimum values of Nu_v). The collision occurs at $\lambda_c \approx 147.6$ and results in two small families and a united family formed by almost all parts of the primary families merged together (see Fig. 3d). Here, saddle bifurcation in terms of the classification of [2, 12, 13] is observed. It should be noted that the secondary-family branch that belongs to the invariant subspace (the segment in the line $\mathrm{Nu}_v = 0$) is not shown in Fig. 3.

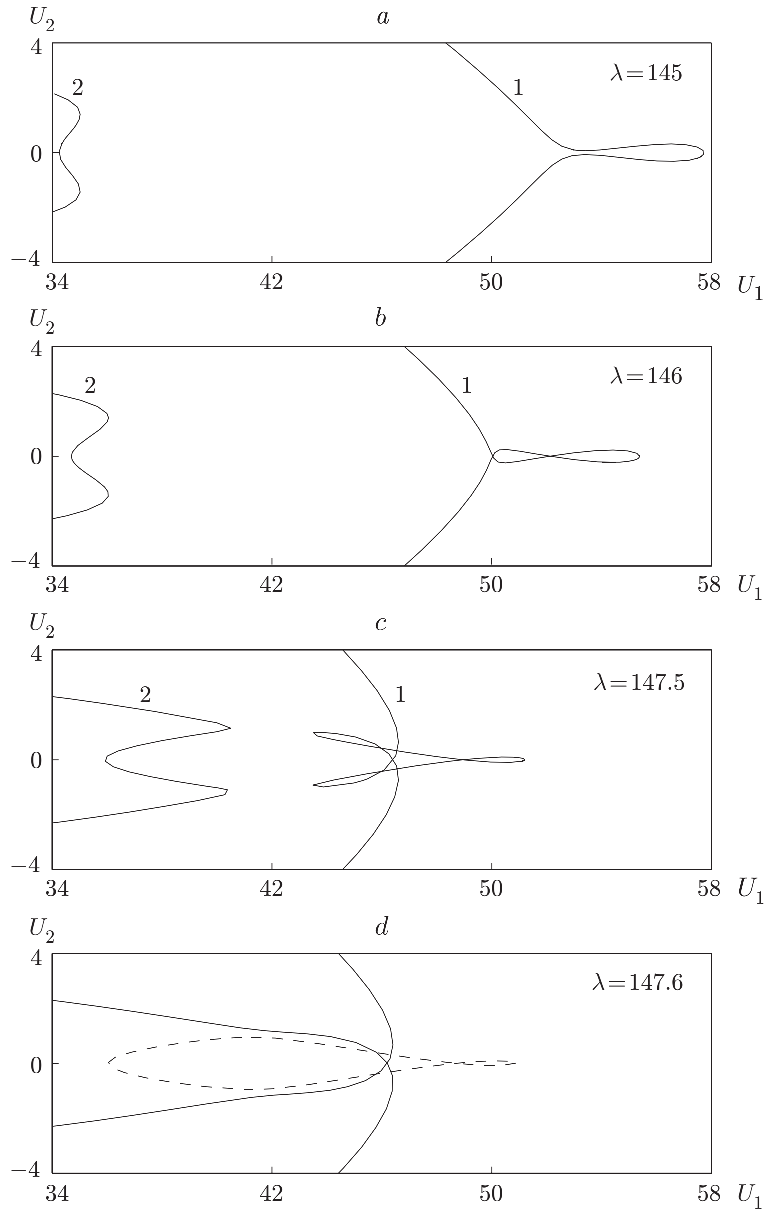

Fig. 4. Approachment (a–c) and subsequent collision (d) of families ($\beta = 2$): 1) primary family, 2) secondary family; the small family departed during the collision is shown by the dashed curve.

Figure 4 depicts the approachment and subsequent collision of the primary and secondary families (curves 1 and 2, respectively). The curves in Fig. 4 are projections onto the manifold formed by zero-equilibrium eigenvectors that correspond to two eigennumbers with the maximum real part. At $\lambda = 156$, the united family becomes absolutely unstable and, as the parameter λ further increases, the small families shrink and vanish, which corresponds to origin bifurcation of the equilibrium cycle "from nowhere" described in [2, 12, 13].

Figure 5 shows the evolution of the families for $\beta = 2.5$. The primary family of equilibria branches off at $\lambda = 46.9$, and at $\lambda = \lambda_u = 106$ the family displays oscillatory instability at four points (see Fig. 5a), from which four unstable-equilibrium arcs emerge as the value of λ further increases. With increasing λ, arcs consisting of the equilibria that lost their stability monotonically appear (Fig. 5b). At $\lambda_o = 122$, the family becomes absolutely unstable, and the trajectories emitted from the vicinity of the family are attracted to the stable limiting cycle that exists at $\lambda > 119$. Figure 5c shows the families prior to the collision, and Fig. 5d shows the result of the collision, namely, the formation of the united family (curve 1) and two small families (curves 2 and 3).

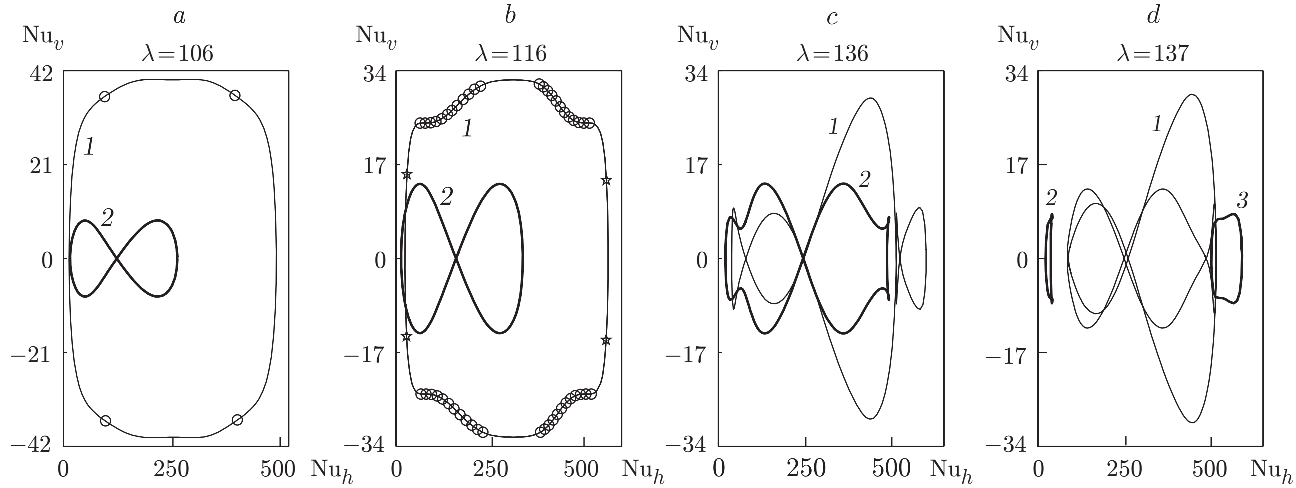

Fig. 5. Families of stationary regimes ($\beta = 2.5$) (notation the same as in Fig. 3).

Calculations with $\beta \geqslant 2.3$ reveal simultaneous existence of stable stationary motions on the family and stable periodic autooscillatory regimes; for $\beta < 2.3$, complex stochastic regimes are found. For instance, after the family of equilibria for a container with $\beta = 2$ becomes absolutely unstable, a chaotic regime emerges ($\lambda = 156$). As λ becomes greater than $\lambda = 164$, the chaotic regime transforms into the limiting cycle; as the parameter λ decreases, this chaotic regime breaks down, and establishment of two stable regimes on the equilibrium family is observed.

The range of filtration Rayleigh numbers in which stable equilibria on the family and stable limiting cycles can exist simultaneously widens with increasing β above $\beta \geqslant 2.3$.

Conclusions. The present study of families of stationary regimes in the filtration convection problem with the Darcy law of friction has allowed us to trace the evolution of primary and secondary families branching off from the zero equilibrium up to their collision for rectangular containers with various height-to-width ratios. Critical values of the filtration Rayleigh number are found at which instability emerges on the primary family, i.e., the second transition occurs. It is shown how the character of instability (oscillatory or monotonic) and the total number of stationary motions losing their stability depend on the container height.

The collision of families of stationary regimes is considered. It is shown that the reconnection of branches of colliding families gives rise to new families, which consist of both stable and unstable stationary regimes. Further studies are required to establish which of these regimes can be observed experimentally. (Some approaches to this problem and analytical and numerical results for the finite-dimensional Darcy problem can be found in [14].)

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